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STABILITY OF SOME SIMPLE MODELS OF TURBULENCE

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Abstract. We present stability results on Markov semigroups generated by simplified stochastic models appearing in hydrodynamics. We consider the Goy and the Sabra shell model with degenerate noise. The models capture some essential statistical properties of turbulent flows. Using the lower bound technique developed in [2, 13] we prove the existence of an invariant measure and its stability.

1. Introduction

The paper is aimed at proving asymptotic stability of Markov processes generated by stochastic hydrodynamical models (the Goy and the Sabra shell model) which are very popular examples of simplified phenomenological models of turbulence. Although they are not based on conservation laws, they capture some essential statistical properties and features of turbulent flows like the energy and the enstrophy cascade and the power law decay of the structure functions in some range of wave numbers, the inertial range. We refer the reader to [1, 5, 9, 10, 20] and the references therein and to [4, 6, 8] for some rigorous results.

We are interested in a Wiener noise disturbance with only finitely many nontrivial modes and then we prove the e-process property. It is possible that the simillar results may be obtained using coupling methods (see for instance [3, 11, 14]). Here we make use of general results on stability of processes satisfying the so-called e–property (see [14, 15]) developed by lower bound technique. In particular, it is known that any Markov process with the e–property which is averagely bounded and with positive probability enters into any neighbourhood of a fixed point is asymptotically stable. In particular, it admits a unique invariant measure. The proof of this result was given in [2]. The most difficult part of proving that the models under consideration satisfy sufficient conditions of our criterion is the proof of the e–property. For its verification we use the Malliavin calculus developed in [12]. The main result of the paper answers to the conjecture posed by Barbato *et al.* (see [4]) who anticipated that in the case when the number of modes to which we add the noise is large enough, it would be possible to prove the uniqueness of an invariant measure.

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The paper is organized as follows. In section 2, we introduce the concepts of e-property, averagely bounded and concentrating at a point. We also formulate the results about asymptotic stability for Markov processes. In Section 3, we introduce the GOY and the Sabra model and give general results about their well posedness. In Section 4, we apply the results of Section 2 to the shell models and prove the e-property, the average boundedness and the concentrating property. Then we state our main result for the uniqueness of the invariant measure for the stochastic shell model with a degenerate noise.

2. Criterion on Stability

Let (X, ρ) be a Polish space. By $B_b(X)$ we denote the space of all bounded Borel–measurable functions equipped with the supremum norm. Let $(P_t)_{t\geq 0}$ be the *Markovian semigroup* defined on $B_b(X)$. For each $t \geq 0$ we have $P_t\mathbf{1}_X = \mathbf{1}_X$ and $P_t\psi \geq 0$ if $\psi \geq 0$. Throughout this paper we shall assume that the semigroup is *Feller*, i.e. $P_t(C_b(X)) \subset C_b(X)$ for all $t > 0$. Here and in the sequel $C_b(X)$ is the subspace of all bounded continuous functions with the supremum norm $\|\cdot\|_\infty$. By $L_b(X)$ we will denote the subspace of all bounded Lipschitz functions. We shall also assume that $(P_t)_{t\geq 0}$ is *stochastically continuous*, which implies that $\lim_{t\to 0^+} P_t\psi(x) = \psi(x)$ for all $x \in X$ and $\psi \in C_b(X)$.

Let $\mathcal{M}_1$ stand for the space of all Borel probability measures on X. Denote by $\mathcal{M}_1^W$, $W \subset X$, the subspace of all Borel probability measures supported in W, i.e. $\{x \in X : \mu(B(x, r)) > 0 \text{ for any } r > 0\} \subset W$, where $B(x, r)$ denotes the ball in X with center at x and radius r. For $\varphi \in B_b(X)$ and $\mu \in \mathcal{M}_1$ we will use the notation $\langle\varphi, \mu\rangle = \int_X \varphi(x)\mu(\mathrm{d}x)$. Recall that the *total variation norm* of a finite signed measure $\mu \in \mathcal{M}_1 - \mathcal{M}_1$ is given by $\|\mu\|_{TV} = \mu^+(X) + \mu^-(X)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

We say that $\mu_* \in \mathcal{M}_1$ is *invariant* for $(P_t)_{t\geq 0}$ if $\langle P_t\psi, \mu_*\rangle = \langle\psi, \mu_*\rangle$ for every $\psi \in B_b(X)$ and $t \geq 0$. Alternatively, we can say that $P_t^*\mu_* = \mu_*$ for all $t \geq 0$, where $(P_t^*)_{t\geq 0}$ denotes the semigroup dual to $(P_t)_{t\geq 0}$, i.e. for a given Borel measure μ, Borel subset A of X, and $t \geq 0$ we set

$$P_t^*\mu(A) := \langle P_t\mathbf{1}_A, \mu\rangle.$$

A semigroup $(P_t)_{t\geq 0}$ is said to be *asymptotically stable* if there exists an invariant measure $\mu_* \in \mathcal{M}_1$ such that $P_t^*\mu$ converges weakly to μ_* as $t \to +\infty$ for every $\mu \in \mathcal{M}_1$. Obviously μ_* is unique.

Definition 2.1. We say that a semigroup $(P_t)_{t\geq 0}$ has *the e–property* if the family of functions $(P_t\psi)_{t\geq 0}$ is equicontinuous at every point x of X for any bounded and Lipschitz function ψ, i.e.

$$\forall\, \psi \in L_b(X),\ x \in X,\ \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in B(x, \delta),\ t \geq 0 :\ |P_t\psi(x) - P_t\psi(z)| < \varepsilon.$$

Remark. One can show (see [13]) that to obtain the e–property in the case when X is a Hilbert space, it is enough to verify the above condition for every function with bounded Fréchet derivative.

Definition 2.2. A semigroup $(P_t)_{t\geq 0}$ is called *averagely bounded* if for any $\varepsilon > 0$ and bounded set $A \subset X$ there is a bounded Borel set $B \subset X$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B)\mathrm{d}s > 1 - \varepsilon \qquad \text{for } \mu \in \mathcal{M}_1^A.$$

Definition 2.3. A semigroup $(P_t)_{t\geq 0}$ is *concentrating* at z if for any $\varepsilon > 0$ and bounded set $A \subset X$ there exists $\alpha > 0$ such that for any two measures $\mu_1, \mu_2 \in \mathcal{M}_1^A$ holds

$$P_t^*\mu_i(B(z, \varepsilon)) \geq \alpha \text{ for } i = 1, 2 \text{ and some } t > 0.$$

In [2] we formulated and proved the following theorems devoted to stability of Markov semigroups with the Feller property. Generally speaking the theorems allege that relevant concentrating conditions imply existence of an invariant measure and its stability. They used the so-called lower bound technique developed by A. Lasota and J. Yorke in [16].

Theorem 2.4. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then for any $\varphi \in L_b(X)$ and $\mu_1, \mu_2 \in \mathcal{M}_1$ we have*

$$\lim_{t\to\infty} |\langle \varphi, P_t^* \mu_1\rangle - \langle \varphi, P_t^* \mu_2\rangle| = 0. \tag{2.1}$$

Theorem 2.5. *Assume that there exists $z \in X$ such that for any $\varepsilon > 0$*

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P_s^*\mu(B(z,\varepsilon))\mathrm{d}s > 0. \tag{2.2}$$

If $(P_t)_{t\geq 0}$ satisfies the e–property, then it admits an invariant measure.

Theorem 2.6. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then it is asymptotically stable.*

3. THE MODELS

3.1. **GOY and Sabra shell models and functional setting.** Let $u = (u_{-1}, u_0, u_1, \ldots)$ be an infinite sequence of complex valued functions on $[0,\infty)$ satisfying the following equations for $n = 1, 2, \ldots$

$$\mathrm{d}u_n(t) + \nu k_n^2 \nu_n(t)\mathrm{d}t + [B(u,u)]_n \mathrm{d}t = \sigma_n \mathrm{d}w_n \tag{3.1}$$

with the initial conditions

$$u_{-1}(t) = u_0(t) = 0 \quad \text{and} \quad u_n(0) = \xi_n.$$

Here $k_n = k_0 2^n$, $k_0 > 1$ and $\nu > 0$. Moreover $(w_n(t))_{n\geq 1}$ denotes a sequence of independent Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that $\sigma_n \in \mathbb{C}$ and there is $n_0 \in \mathbb{N}$ such that $\sigma_n = 0$ for $n \geq n_0$. Further B is a bilinear operator which will be defined later on.

Let H be the set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider H as a *real* Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$ of the form

$$(u,v) = \mathrm{Re} \sum_{n\geq 1} u_n v_n^*, \quad |u|^2 = \sum_{n\geq 1} |u_n|^2, \tag{3.2}$$

where v_n^* denotes the complex conjugate of v_n. The space H is separable. Let $A : \mathrm{D}(A) \subset H \to H$ be the non-bounded linear operator defined by

$$(Au)_n = k_n^2 u_n, \quad n = 1, 2, \ldots, \qquad \mathrm{D}(A) = \Big\{u \in H \,:\, \sum_{n\geq 1} k_n^4 |u_n|^2 < \infty\Big\}.$$

The operator A is clearly self-adjoint, strictly positive definite since $(Au, u) \geq k_0^2|u|^2$ for $u \in \mathrm{D}(A)$. For any $\alpha > 0$, set

$$\mathcal{H}_\alpha = \mathrm{D}(A^\alpha) = \{u \in H \,:\, \sum_{n\geq 1} k_n^{4\alpha}|u_n|^2 < +\infty\},\ \|u\|_\alpha^2 = \sum_{n\geq 1} k_n^{4\alpha}|u_n|^2 \text{ for } u \in \mathcal{H}_\alpha.$$

Obviously $\mathcal{H}_0 = H$. Define

$$V := \mathrm{D}(A^{\frac{1}{2}}) = \Big\{ u \in H \,:\, \sum_{n\ge 1} k_n^2 |u_n|^2 < +\infty \Big\}$$

and set

$$\mathcal{H} = \mathcal{H}_{\frac{1}{4}},\ \|u\|_{\mathcal{H}} = \|u\|_{\frac{1}{4}}.$$

Then V is a Hilbert space for the scalar product $(u,v)_V = \mathrm{Re}(\sum_n k_n^2\, u_n\, v_n^*)$, $u,v \in V$ and the associated norm is denoted by

$$\|u\|^2 = \sum_{n\ge 1} k_n^2\, |u_n|^2.$$

The adjoint of V with respect to the H scalar product is $V' = \{(u_n) \in \mathbb{C}^{\mathbb{N}} \,:\, \sum_{n\ge 1} k_n^{-2}\, |u_n|^2 < +\infty\}$ and $V \subset H \subset V'$ is a Gelfand triple. Let $\langle u\,,\,v\rangle_{V',V} = \mathrm{Re}\left(\sum_{n\ge 1} u_n\, v_n^*\right)$ denote the duality between $u \in V'$ and $v \in V$.

Set $u_{-1} = u_0 = 0$, let a, b be real numbers and let $B : H \times V \to H$ (or $B : V \times H \to H$) denote the bilinear operator defined by

$$[B(u,v)]_n = i\left(a k_{n+1} u_{n+1}^* v_{n+2}^* + b k_n u_{n-1}^* v_{n+1}^* - a k_{n-1} u_{n-1}^* v_{n-2}^* - b k_{n-1} u_{n-2}^* v_{n-1}^*\right)$$

for $n = 1, 2, \ldots$ in the GOY shell model (see, e.g. [20]) or

$$[B(u,v)]_n = i\left(a k_{n+1} u_{n+1}^*\, v_{n+2} + b k_n u_{n-1}^* v_{n+1} + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} u_{n-2} v_{n-1}\right),$$

in the Sabra shell model introduced in [17].

Obviously, there exists $C > 0$ such that

$$|B(u,v)| \le C\|u\||v| \qquad \text{for } u \in V \text{ and } v \in H. \tag{3.3}$$

Note that B can be extended as a bilinear operator from $H \times H$ to V' and that there exists a constant $C > 0$ such that given $u, v \in H$ and $w \in V$ we have

$$|\langle B(u,v)\,,\,w\rangle_{V',V}| + |\big(B(u,w)\,,\,v\big)| + |\big(B(w,u)\,,\,v\big)| \le C\,|u|\,|v|\,\|w\|. \tag{3.4}$$

An easy computation proves that for $u, v \in H$ and $w \in V$ (resp. $v, w \in H$ and $u \in V$),

$$\langle B(u,v)\,,\,w\rangle_{V',V} = -\big(B(u,w)\,,\,v\big) \text{ (resp. } \big(B(u,v)\,,\,w\big) = -\big(B(u,w)\,,\,v\big) \text{).}$$

Hence $(B(v,u),u) = 0$ for $u \in H$ and $v \in V$. Furthermore, $B : V \times V \to V$ and $B : \mathcal{H} \times \mathcal{H} \to H$; indeed, for $u, v \in V$ (resp. $u, v \in \mathcal{H}$) we have

$$\|B(u,v)\|^2 = \sum_{n\ge 1} k_n^2\, |B(u,v)_n|^2 \le C\,\|u\|^2 \sup_n k_n^2 |v_n|^2 \le C\,\|u\|^2\,\|v\|^2,$$
$$|B(u,v)| \le C\,\|u\|_{\mathcal{H}}\,\|v\|_{\mathcal{H}}.$$

3.2. **Well-posedness.** Consider the abstract equation on H of the form

$$\mathrm{d}u(t) = [-\nu A u(t) + B(u(t),u(t))]\,\mathrm{d}t + Q\mathrm{d}W(t), \quad t \ge 0 \tag{3.5}$$

with the initial condition $u(0) = \xi \in H$, where $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ is some matrix with $\mathrm{Tr}(QQ^*) < \infty$ and $W(t) = (w_n(t))_{n\ge 1}$ is a cylindrical Wiener noise on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

Definition 3.1. A stochastic process $u(t,\omega)$ is a generalized solution in $[0,T]$ of the system (3.5) if

$$u(\cdot,\omega)\in C([0,T];H)\cap L^2(0,T;\mathcal{H})$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$, u is progressively measurable in these topologies and equation (3.5) is satisfied in the integral sense

$$\begin{aligned}&(u(t),\varphi)+\int_0^t \nu(u(s),A\varphi)\mathrm{d}s+\int_0^t\left(B\left(u(s),\varphi\right),u(s)\right)\mathrm{d}s\\&=(\xi,\varphi)+(QW(t),\varphi)\end{aligned}$$

for all $t\in[0,T]$ and $\varphi\in \mathrm{D}(A)$.

Theorem 3.2. *Let us assume that the initial condition ξ is an $\mathcal{F}_0$-random variable with values in H. Then there exists a unique solution $(u(t))_{t\geq 0}$ to equation* (3.5)*. Moreover, if $\mathbb{E}|\xi|^2<+\infty$, then*

$$\mathbb{E}|u(t)|^2+\int_0^t 2\nu\mathbb{E}\|u(s)\|^2\mathrm{d}s=\mathbb{E}|\xi|^2+\mathrm{Tr}(QQ^*)t \tag{3.6}$$

for any $t\geq 0$.

Proof. We will prove well–posedness using a pathwise argument (for similar results see [4] and the references therein). Let us introduce the Ornstein-Uhlenbeck process solution of

$$\begin{cases}\mathrm{d}z(t)+\nu Az(t)\mathrm{d}t=Q\mathrm{d}W,\\ z(0)=0.\end{cases} \tag{3.7}$$

The above equation has a unique progressively measurable solution such that $\mathbb{P}$-a.s.

$$z\in C([0,T];\mathcal{H})$$

(for more details see [7]). Set $v=u-z$. Then for $\mathbb{P}$-a.e. $\omega\in\Omega$

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}v(t)+\nu Av(t)-B(v(t)+z(t),v(t)+z(t))=0,\\ v(0)=\xi,\end{cases} \tag{3.8}$$

is a deterministic system. The existence and uniqueness of global weak solutions v follow from the Galerkin approximation procedure and then passing to the limit using the appropriate compactness theorems. We omit the details which can be found in [4] and the references therein. Instead, we present the formal computations which lead to the basic a priori estimates, this is in order to stress the role played by z. Using equation (3.8) and various properties of the nonlinear operator B, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v(t)|^2+\nu\|v(t)\|^2 &\leq |(B(v(t)+z(t),z(t)),v(t))|\\ &\leq C\|v(t)\||v(t)+z(t)||z(t)|\\ &\leq \frac{\nu}{2}\|v(t)\|^2+C(\nu)\left(|v(t)|^2|z(t)|^2+|z(t)|^4\right).\end{aligned}$$

Using Gronwall's Lemma and the fact that $\|z\|_{C([0,T];\mathcal{H})}\leq C(\omega)$, we have

$$\sup_{0\leq t\leq T}|v(t)|^2\leq C(|\xi|,T,C(\omega)).$$

Again, using the above inequality in the previous estimate, we obtain that

$$\int_0^T \|v(s)\|^2 \mathrm{d}s \le C(|\xi|, T, C(\omega)).$$

Then, by classical arguments, see [21], $v \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/2}))$. Therefore $u = v + z \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/4}))$ $\mathbb{P}$-a.s.

To finish the proof observe that condition (3.6) follows from Itô's formula. □

The uniqueness of solutions is established in the following theorem.

Theorem 3.3. *Let $\left(u^{(1)}(t)\right)_{t\ge 0}$, $\left(u^{(2)}(t)\right)_{t\ge 0}$, be two continuous adapted solutions of* (3.5) *in H, with the initial conditions $u_0^{(1)}$ and $u_0^{(2)}$ as above. Then there is a constant $C(\nu) > 0$, depending only on ν, such that $\mathbb{P}$-a.s.*

$$\left|u^1(t) - u^2(t)\right|^2 \le \mathrm{e}^{C(\nu)\int_0^t \left|u^1(s)\right|^2 ds} \left|u_0^1 - u_0^2\right|^2 \quad t \ge 0.$$

Proof. Let us put $u(t) = u^1(t) - u^2(t)$. Then u is the solution of the following equation

$$\mathrm{d}u + \nu A u \mathrm{d}t - \left(B(u^1,u^1) - B(u^2,u^2)\right)\mathrm{d}t = 0.$$

Using again the properties of operator B, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 &\le |(B(u,u^1),u)| \\ &\le \frac{\nu}{2}\|u\|^2 + C(\nu)|u|^2|u^1|^2. \end{aligned}$$

Hence, by the Gronwall lemma, we obtain that

$$|u(t)|^2 \le |u(0)|^2 \mathrm{e}^{C(\nu)\left(\int_0^T |u^1(s)|^2 ds\right)},$$

which finishes the proof. □

4. Stability of the model

Let a diagonal matrix $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ be such that there is $n_0 \in \mathbb{N}$ and $q_{n,n} = 0$ for $n \ge n_0$. Consider the equation on H of the form

$$\mathrm{d}u(t) = [-\nu A u(t) + B(u(t),u(t))]\mathrm{d}t + Q\mathrm{d}W(t) \quad t \ge 0, \tag{4.1}$$

where $(W(t))_{t\ge 0}$ is a certain cylindrical Wiener process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

By Theorem 3.2 for every $x \in H$ there is a unique continuous solution $(u^x(t))_{t\ge 0}$ in H, hence the transition semigroup is well defined. From Theorem 3.3 we obtain that the solution satisfies the Feller property, i.e. for any $t \ge 0$ if $x_n \to x$ in H, then $\mathbb{E}f(u^{x_n}(t)) \to \mathbb{E}f(u^x(t))$ for any $f \in C_b(H)$. Set

$$P_t f(x) = \mathbb{E}f(u^x(t)) \quad \text{for any } f \in C_b(H).$$

Obviously $(P_t)_{t\geq 0}$ is stochastically continuous. First note that $DP_tf(x)[v]$, the value of the Frechet derivative $DP_tf(x)$ at $v\in H$, is equal to $\mathbb{E}\,\{Df(u^x(t))[U(t)]\}$, where $U(t):=\partial u^x(t)[v]$ and

$$\partial u^x(t)[v] := \lim_{\eta\downarrow 0}\frac{1}{\eta}\left(u^{x+\eta v}(t)-u^x(t)\right)$$

and the limit is in $L^2(\Omega,\mathcal{F},\mathbb{P};H)$ (see [13] also [12]). The process $U=(U(t))_{t\geq 0}$ satisfies the linear evolution equation

$$\begin{aligned}\frac{\mathrm{d}U(t)}{\mathrm{d}t} &= -\nu AU(t)+B(u^x(t),U(t))+B(U(t),u^x(t)),\\ U(0)&=v.\end{aligned} \tag{4.2}$$

Suppose that $\mathcal{X}$ is a certain Hilbert space and $\Phi\colon H\to\mathcal{X}$ a Borel measurable function. Given an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $g\colon[0,\infty)\times\Omega\to H$ satisfying $\mathbb{E}\int_0^t|g(s)|^2\mathrm{d}s<\infty$ for each $t\geq 0$ we denote by $\mathcal{D}_g\Phi(u^x(t))$ the Malliavin derivative of $\Phi(u^x(t))$ in the direction of g; that is the $L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{X})$-limit, if exists, of

$$\mathcal{D}_g\Phi(u^x(t)) := \lim_{\eta\downarrow 0}\frac{1}{\eta}\left[\Phi(u^x_{\eta g}(t))-\Phi(u^x(t))\right],$$

where $u^x_g(t)$, $t\geq 0$, solves the equation

$$\mathrm{d}u^x_g(t)=\left[-\nu Au^x_g(t)+B(u^x_g(t),u^x_g(t))\right]\mathrm{d}t+Q\left(\mathrm{d}W(t)+g(t)\mathrm{d}t\right),\qquad u^x_g(0)=x.$$

In particular, one can easily show that when $\mathcal{X}=H$ and $\Phi=I$, where I is the identity operator, the Malliavin derivative of $u^x(t)$ exists and the process $D(t):=\mathcal{D}_gu^x(t)$, $t\geq 0$, solves the linear equation

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t}(t) &= -\nu AD(t)+B(u^x(t),D(t))+B(D(t),u^x(t))+Qg(t),\\ D(0)&=0.\end{aligned} \tag{4.3}$$

Directly from the definition of the Malliavin derivative we conclude the *chain rule:* suppose that $\Phi\in C^1_b(H;\mathcal{X})$ then

$$\mathcal{D}_g\Phi(u^x(t))=D\Phi(u^x(t))[D(t)].$$

(Here $C^1_b(H;\mathcal{X})$ denotes the space of all bounded continuous functions $\Phi:H\to\mathcal{X}$ with continuous and bounded first derivative with the natural norm. In the case when $\mathcal{X}=\mathbb{R}$ we simply write $C^1_b(H)$.) In addition, the *integration by parts formula* holds, see Lemma 1.2.1, p. 25 of [18]. Indeed, suppose that $\Phi\in C^1_b(H)$. Then

$$\mathbb{E}[\mathcal{D}_g\Phi(u^x(t))]=\mathbb{E}\left[\Phi(u^x(t))\int_0^t(g(s),\mathrm{d}W(s))\right]. \tag{4.4}$$

Lemma 4.1. *Let* $\eta\in(0,\nu/(2\max q^2_{i,i})]$. *Then we have*

$$\mathbb{E}(\exp(\eta|u^x(t)|^2+\eta\nu\int_0^t\|u^x(s)\|^2\mathrm{d}s))\leq 2\exp(\eta(\operatorname{Tr}Q^2)t+\eta|x|^2).$$

Proof. Fix $\eta \in (0, \nu/(2\max q_{i,i}^2)]$. Let $M(t) = \eta\int_0^t (u^x(s), Q\mathrm{d}W(s))$ and let $N(t) = M(t) - \eta\nu\int_0^t \|u^x(s)\|^2\mathrm{d}s$. Set $\alpha = \nu/\max q_{i,i}^2$. Then we have $\nu\|u^x(s)\|^2 \geq \alpha|Qu^x(s)|^2$. Now observe that $N(t) \leq M(t) - (\alpha/\eta)\langle M\rangle(t)$, where $\langle M\rangle(t)$ denotes the quadratic variation of the continous L^2–martingale M with the filtration generated by the noise. Hence by a standard variation of the Kolmogorov–Doob martingale inequality (see [19]) we have

$$\mathbb{P}(N(t) \geq K) \leq \exp(-\alpha K/\eta)$$

and consequently we obtain

$$\mathbb{P}(\exp N(t) \geq \exp K) \leq \exp(-\alpha K/\eta) \leq \exp(-2K)$$

for any $K > 0$. An easy observation that if some positive random variable, say X, satisfies the condition $\mathbb{P}(X \geq C) \leq C^{-2}$ for every $C > 0$, then $\mathbb{E}X \leq 2$ gives

$$\mathbb{E}(\exp(\eta|u^x(t)|^2 + \eta\nu\int_0^t \|u^x(s)\|^2\mathrm{d}s - \eta(\mathrm{Tr}\,Q^2)t - \eta|x|^2)) \leq 2,$$

by Itô's formula. This completes the proof. □

The crucial role in our consideration is played by the following lemma. The idea of its proof is taken from [12].

Lemma 4.2. *Let* $(P_t)_{t\geq 0}$ *correspond to problem* (4.1). *If* Q *satisfies the condition:*

$$q_{1,1}, \ldots, q_{N_*,N_*} \neq 0 \quad \text{for } N_* > \log_2(2C^2\max q_{i,i}^2/\nu^3 + \mathrm{Tr}\,Q^2/(2\max q_{i,i}^2))/2, \tag{4.5}$$

where $C > 0$ *is given by (3.3), then for any* $f \in C_b^1(H)$ *and* $R > 0$ *there exists a constant* $C_0 > 0$ *such that*

$$\sup_{t\geq 0}\sup_{|x|\leq R}\sup_{|v|\leq 1}|DP_tf(x)[v]| \leq C_0\|f\|_{C_b^1(H)}. \tag{4.6}$$

Proof. Fix $N_* > \log_2(2C^2\max q_{i,i}^2/\nu^3 + \mathrm{Tr}\,Q^2/(2\max q_{i,i}^2))/2$. The proof will be split into three steps.

Step I: Let $g : [0,\infty)\times\Omega \to H$ be a measurable function such that $\mathbb{E}\int_0^t |g(s)|^2\mathrm{d}s < \infty$ for any $t \geq 0$. Let $\omega_t(x) := \mathcal{D}_g u^x(t)$ and $\rho_t(v,x) := \partial u^x(t)[v] - \mathcal{D}_g u^x(t)$. Then,

$$\begin{aligned} DP_tf(x)[v] &= \mathbb{E}\{Df(u^x(t))[\omega_t(x)]\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\} \\ &= \mathbb{E}\{\mathcal{D}_g f(u^x(t))\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\} \\ &\overset{(4.4)}{=} \mathbb{E}\left\{f(u^x(t))\int_0^t (g(s),\mathrm{d}W(s))\right\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\}. \end{aligned}$$

We have

$$\left|\mathbb{E}\left\{f(u^x(t))\int_0^t (g(s),\mathrm{d}W(s))\right\}\right| \leq \|f\|_{L^\infty}\left(\mathbb{E}\int_0^t |g(s)|^2\mathrm{d}s\right)^{1/2}$$

and

$$|\mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\}| \leq \|f\|_{C_b^1(H)}\mathbb{E}|\rho_t(v,x)| \leq \|f\|_{C_b^1(H)}(\mathbb{E}|\rho_t(v,x)|^2)^{1/2}.$$

Step II: Let $\xi(t) = (\xi_1(t), \xi_2(t), \ldots) : [0,\infty) \to H$ be a solution to the following system:

$$\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} = -\frac{\xi_i(t)}{2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)}} \qquad \text{for } i = 1,\ldots,N_*$$

$$\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} = -\nu k_i^2\xi_i(t) + [B(u^x(t),\xi(t)) + B(\xi(t),u^x(t))]_i \qquad \text{for } i \geq N_* + 1.$$

with $\xi(0) = v$. We assume also that $\xi_i(t)/2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)} = 0$ if $\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)} = 0$ (see [12]). Observe that $\xi_1(t), \xi_2(t), \dots, \xi_{N_*}(t) = 0$ for $t \geq 2$.

Now we choose $g : [0, +\infty) \times \Omega \to H$ to be given by the formulae:

$$g_i(t) = \frac{1}{q_{i,i}}\left(-\nu k_i^2 \xi_i(t) + [B(u^x(t), \xi(t)) + B(\xi(t), u^x(t))]_i - \frac{\xi_i(t)}{2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)}}\right)$$

for $i = 1, \dots, N_*$ and $g_i(t) = 0$ for $i \geq N_* + 1$.

It is easy to see that $\rho_t = \xi(t)$ for any $t \geq 0$. Indeed, observe that

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} + Qg(t) = -\nu A\xi(t) + B(u^x(t), \xi(t)) + B(\xi(t), u^x(t))$$

and

$$\xi(0) = v.$$

On the other hand, subtracting equation (4.2) from (4.3) we obtain the equation for ρ_t. Since ρ_t and $\xi(t)$ solve the same equation with the same initial condition $\rho_0 = \xi(0) = v$, we obtain $\rho_t = \xi(t)$ for $t \geq 0$.

Step III: To show (4.6) it is enough to prove that

$$\sup_{|x|\leq R}\sup_{|v|\leq 1}\mathbb{E}\int_0^\infty |g(s)|^2\mathrm{d}s < \infty$$

and

$$\sup_{t\geq 0}\sup_{|x|\leq R}\sup_{|v|\leq 1}\mathbb{E}|\xi(t)|^2 < \infty.$$

We know that $\sum_{i=1}^{N_*}|\xi_i(t)|^2 \leq |v|^2 \leq 1$ for $t \geq 0$. In particular $\xi_i(t) = 0$ for $t \geq 2$ and $i = 1, \dots, N_*$. Let $\zeta(t) = (\xi_{N_*+1}(t), \xi_{N_*+2}(t), \dots)$. It is easy to see that ζ satisfies the inequality

$$\frac{\mathrm{d}|\zeta(t)|^2}{\mathrm{d}t} \leq -\nu k_{N_*}^2|\zeta(t)|^2 + 2C\|u^x(t)\||\zeta(t)|^2 + 2\tilde{C}\|u^x(t)\||\zeta(t)| \quad \text{for } t \geq 0, \tag{4.7}$$

where $\tilde{C}$ is some positive constant dependent only on C. Choose $\varepsilon > 0$ and $\gamma \in (0, 1)$ such that

$$-\nu k_{N_*}^2 + \varepsilon + 2C^2\max q_{i,i}^2/\nu^2 + \nu\operatorname{Tr}Q^2/(2\gamma\max q_{i,i}^2) < 0.$$

From equation (4.7) we derive

$$\frac{\mathrm{d}|\zeta(t)|^2}{\mathrm{d}t} \leq (-\nu k_{N_*}^2 + 2C\|u^x(t)\| + \varepsilon)|\zeta(t)|^2 + C(\varepsilon)\|u^x(t)\|^2$$

and using Gronwall's lemma we obtain

$$\begin{aligned}|\zeta(t)|^2 &\leq \left(|v|^2 + C(\varepsilon)\int_0^t \|u^x(s)\|^2\mathrm{d}s\right)e^{(-\nu k_{N_*}^2+\varepsilon)t + 2C\int_0^t\|u^x(s)\|\mathrm{d}s} \\ &\leq e^{(-\nu k_{N_*}^2+\varepsilon)t}\left[1 + C(\varepsilon)\int_0^t\|u^x(s)\|^2\mathrm{d}s\right]e^{2C\int_0^t\|u^x(s)\|\mathrm{d}s}.\end{aligned}$$

Hence we obtain that there exist constant $A > 0$ (independent of $t \geq 0$, $v \in B(0, 1)$ and $x \in B(0, R)$) such that

$$|\zeta(t)|^2 \leq A\exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2\max q_{i,i}^2/\nu^2)t)\exp\left(\nu/(2\max q_{i,i}^2)\int_0^t\|u^x(s)\|^2\mathrm{d}s\right)$$

for all $t \geq 0$, by the fact that $-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 < 0$. Thus

$$\sup_{|x|\leq R, |v|\leq 1, t\geq 0} \mathbb{E}|\zeta(t)|^2$$
$$\leq A \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)t)\mathbb{E}\left(\exp\left(\nu/(2\max q_{i,i}^2)\int_0^t \|u^x(s)\|^2 \mathrm{d}s\right)\right).$$

Using Lemma 4.1 we obtain

$$\sup_{t\geq 0, |x|\leq R, |v|\leq 1} \mathbb{E}|\zeta(t)|^2 \leq \tilde{A}\exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu/(2\gamma \max q_{i,i}^2) \operatorname{Tr} Q^2)t)$$

for some $\tilde{A} > 0$. On the other hand, by the definition of N_*, k_n and the choice of ε, γ we have

$$-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu/(2\gamma \max q_{i,i}^2) \operatorname{Tr} Q^2 < 0.$$

Now we must evaluate

$$\mathbb{E}\int_0^t |g(s)|^2 \mathrm{d}s \leq 2 \sup_{0\leq s\leq 2} \mathbb{E}|g(s)|^2 + \mathbb{E}\int_2^t |g(s)|^2 \mathrm{d}s.$$

The first term on the right side of the above inequality is bounded uniformly in $|x| \leq R$ and $|v| \leq 1$. Further, for $s \geq 2$ we have

$$|g(s)| \leq \tilde{C}\|u^x(s)\||\zeta(s)|$$

and

$$\mathbb{E}\int_2^t |g(s)|^2 \mathrm{d}s \leq \tilde{C}^2 \mathbb{E}\int_2^\infty \|u^x(s)\|^2 |\zeta(s)|^2 \mathrm{d}s$$
$$\leq \hat{C}\,\mathbb{E}[\int_2^\infty \|u^x(s)\|^2 \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s)$$
$$\times \exp(\nu/(2\max q_{i,i}^2)\int_0^s \|u^x(r)\|^2 \mathrm{d}r)\mathrm{d}s]$$
$$\leq \hat{C}\mathbb{E}[\int_2^\infty \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s)$$
$$\times \exp(\nu/(2\max q_{i,i}^2)|u^x(s)|^2 + \nu/(2\max q_{i,i}^2)\int_0^s \|u^x(r)\|^2 \mathrm{d}r)\mathrm{d}s]$$
$$\leq \hat{C}\int_2^\infty [\exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s)$$
$$\times \mathbb{E}\exp(\nu/(2\max q_{i,i}^2)|u^x(s)|^2 + \nu/(2\max q_{i,i}^2)\int_0^s \|u^x(r)\|^2 \mathrm{d}r)]\mathrm{d}s$$
$$\leq C'\int_2^\infty \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu \operatorname{Tr} Q^2/(2\gamma \max q_{i,i}^2))s)\mathrm{d}s,$$

for any $x \in B(0, R)$, where the constant C' depends only on R. Using again the assumption on N_* we obtain

$$\sup_{|x|\leq R, |v|\leq 1} \mathbb{E}\int_2^\infty |g(s)|^2 \mathrm{d}s < \infty.$$

This completes the proof. □

Lemma 4.3. *(Average boundedness) Let* $(P_t)_{t\geq 0}$ *correspond to problem* (3.5)*. Then* $(P_t)_{t\geq 0}$ *is averagely bounded.*

Proof. Fix an $\varepsilon > 0$ and let $r > 0$ be given. If $x \in B(0,r)$, then

$$\begin{aligned}\frac{1}{T}\int_0^T P_s^*\delta_x(H \setminus B(0,R))\mathrm{d}s &= \frac{1}{T}\int_0^T \mathbb{P}(|u^x(s)| > R)\mathrm{d}s \le \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\| > R)\mathrm{d}s \\ &= \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\|^2 > R^2)\mathrm{d}s \le \frac{1}{T}\int_0^T \frac{\mathbb{E}\|u^x(s)\|^2}{R^2}\mathrm{d}s \\ &= \frac{1}{\nu R^2}\frac{1}{T}\int_0^T \nu\mathbb{E}\|u^x(s)\|^2\mathrm{d}s \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + |x|^2/T) \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + r^2/T)\end{aligned}$$

for arbitrary $R > 0$, by (3.6). Hence there is $R_0 > 0$ such that

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0))\mathrm{d}s > 1-\varepsilon.$$

On the other hand, by Fatou's lemma we have

$$\begin{aligned}\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_s^*\mu(B(0,R_0))\mathrm{d}s &\ge \int_H \left(\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0))\mathrm{d}s\right)\mu(\mathrm{d}x) \\ &\ge \int_H (1-\varepsilon)\mu(\mathrm{d}x) = 1-\varepsilon\end{aligned}$$

for any $\mu \in \mathcal{M}_1^{B(0,r)}$. The proof is complete. □

Lemma 4.4. *(Concentrating at* 0*) Let* $(P_t)_{t\ge 0}$ *correspond to problem* (3.5)*. Then* $(P_t)_{t\ge 0}$ *is concentrating at* 0*.*

Proof. Consider first the deterministic equation

$$\mathrm{d}v^x(t) = [-\nu A v^x(t) + B(v^x(t), v^x(t))]\mathrm{d}t$$

with the initial condition $v^x(0) = x$. Then

$$\frac{1}{2}\frac{\mathrm{d}|v^x(t)|^2}{\mathrm{d}t} \le -\nu k_0 |v^x(t)|^2$$

and consequently

$$|v^x(t)|^2 \to 0 \qquad \text{as } t \to +\infty$$

uniformly on bounded sets. Further, fix $\varepsilon > 0$ and $r > 0$. Let $t_0 > 0$ be such that $v^x(t_0) \in B(0,\varepsilon/2)$ for all $x \in B(0,r)$. We may show (see Theorem 8 in [4]) that the process corresponding to the considered model is stochastically stable (see also [13]), i.e. there exists $\eta > 0$ and the set $F_\eta = \{\omega \in \Omega : \sup_{0\le t\le t_0} |QW(t)(\omega)| \le \eta\}$ such that

$$|u^x(t_0)(\omega) - v^x(t_0)| \le \varepsilon/2 \qquad \text{for any } \omega \in F_\eta.$$

Since the process is degenerate, we have $\alpha := \mathbb{P}(F_\eta) > 0$. Consequently, we obtain

$$P_{t_0}^*\delta_x(B(0,\varepsilon)) \ge \mathbb{P}(\{\omega \in \Omega : u^x(t_0)(\omega) \in B(0,\varepsilon)\}) \ge \mathbb{P}(F_\eta) = \alpha$$

for arbitrary $x \in B(0,r)$. Since

$$P_{t_0}^*\mu(B(0,\varepsilon)) = \int_H P_{t_0}^*\delta_x(B(0,\varepsilon))\mu(\mathrm{d}x),$$

we obtain $P_{t_0}^*\mu(B(0,\varepsilon)) \ge \alpha$ for any $\mu \in \mathcal{M}_1^{B(0,r)}$. But $\varepsilon > 0$ and $r > 0$ were arbitrary and hence the concentrating property follows. □

We may formulate the main theorem of this part of our paper.

Theorem 4.5. *The semigroup $(P_t)_{t\geq 0}$ corresponding to problem (3.5) with Q satisfying condition (4.5) is asymptotically stable. In particular, it admits a unique invariant measure.*

Proof. From Lemma 4.2 it follows that the semigroup $(P_t)_{t\geq 0}$ satisfies the e–property. It is also averagely bounded and concentrating at 0, by Lemmas 4.3 and 4.4. Application of Theorem 2.6 finishes the proof. □

Remark: Observe that condition (4.5) implies that the system with not too much noise is stable even when the noise is added to the first mode only.

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